

# The Limits of Logic

## Philosophy 450, Fall 2018

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Tuesday and Thursday 9:30–10:50am, VKC 211

With the tools of formal logic, finite beings can understand the limits on finite beings—and also what is beyond those limits. There are infinities that cannot be counted. For any language, there are properties it cannot express. There are questions that cannot be systematically answered. For any reasonable theory, there are facts that it cannot prove.

### Office Hours

Stonier 227, Tuesday and Thursday 11am–12pm or by appointment

**Please come to office hours** or make an appointment if you feel like you could use help with problems or more discussion of what's going on.

### Goals

You will gain **skills**:

- Reading and understanding precise statements, definitions, and arguments
- Finding careful, step-by-step justifications for abstract claims (“informal proofs”)
- Presenting and explaining technical ideas to others
- Using precise language and reasoning to describe and understand *language and reasoning* themselves

You will gain **knowledge** and **understanding**:

- What are **infinite** collections like? Which infinite collections can be counted using ordinary numbers? Why does this matter?
- How can we give **precise models of language** and its relationship to the world?
- How can we use these models to understand logical ideas like **consistent theories** and **valid arguments**?

- What is a **definable set**? What kinds of features can and cannot be described in a precise language?
- What is an **effectively decidable question**? What kinds of questions can and cannot be answered by any systematic general method?
- What is a **formal proof**? What kinds of statements can and cannot be proved from a reasonable set of assumptions?

You will cultivate **intellectual virtues**:

- **Patience** to stick with a hard complicated problem, going through its pieces slowly until things make sense
- **Care** with the details, to describe things accurately and reason without leaps of logic
- **Confidence** to take on new ideas, even if they are expressed using weird symbols
- **Wonder** at the world of the infinite and our place in it as finite thinking beings

## Evaluation

The only way to learn logic is by doing logic. The best way to understand an idea is to explain it someone else. You'll be responsible not just for taking in the content that I'll teach, but for discovering ideas and teaching them to others.

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<b>Problem presentations</b> See the separate handout.	40%
<b>Participation</b> Be present, be on time, ask questions, and be helpful to others.	5%
<b>Take-home midterm exam</b> Due Thursday, October 2	20%
<b>Take-home final exam</b> Due Tuesday, December 6	35%

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## Course Text

Our main text is a draft of a textbook I am writing, called *The Limits of Logic*. This is available on my website:

<http://www-bcf.usc.edu/~russ813/limits.of.logic.pdf>

This is a work in progress, so while I hope it will be pretty stable, I'm still fixing mistakes and I may need to make some changes as the semester progresses. So if you prefer a printed copy, I don't recommend printing it out more than about a chapter at a time.

I would also love to get your feedback on the textbook! If you find mistakes, parts that are hard to understand, or things you especially like or don't like, this is all really useful for me to know.

## Other Resources

Here are some other suggestions of things to look at, if you'd like to get a different perspective. (If you come across other useful resources, please let me know!)

- Sider, *Logic for Philosophy*
- Boolos, Burgess, and Jeffrey, *Computability and Logic*
- Smith, *An Introduction to Gödel's Theorems*
- Enderton, *Mathematical Logic*
- The Open Logic Project: <http://openlogicproject.org/download/>

Tim Gowers has a series of blog posts on basic logical reasoning for doing informal proofs. They may be helpful for getting the hang of solving the exercises. (But they are aimed at math students, so some of the examples might be kind of alien.)

<https://gowers.wordpress.com/2011/10/09/basic-logic-summary/>

<https://gowers.wordpress.com/category/cambridge-teaching/basic-logic/>

## Outline of Topics

### 1 Sets and Functions

#### Core Skills

- *Informal proofs*. Discovering and presenting precise step-by-step justifications

- *Basic set theory.* Reasoning about relationships between collections of objects
- *One-to-one correspondences.* Using functions to compare the sizes of collections.

#### Key Facts

- *Cantor's Theorem.* Every set has more subsets than it has elements.

## 2 The Infinite and the Uncountable

#### Core Skills

- *Inductive proofs.* We can prove something about all numbers by taking for granted things about smaller numbers.
- *Recursive definitions.* We can build up infinite functions using simple rules.
- *Counting arguments.* We can use counting facts to prove lots of surprising things.

#### Key Facts

- There are just as many finite sequences as counting numbers.
- *Cantor's Theorem Version 2.* There are more infinite sequences than counting numbers.
- There are infinite sets with infinitely many different sizes.

## 3 Structures

#### Core Skills

- *Syntax and semantics.* We can give precise models of simple languages and their relationship to the world.
- *Structural induction and recursion.* We can use simple rules to prove properties and define functions for a whole infinite language.

## Key Facts

- No formal language with finitely many basic symbols can represent every operation in an infinite structure.

## 4 First-Order Logic

### Core Skills

- *More syntax and semantics.* We can give precise models of more complicated languages and their relationship to the world.
- *“Meta-logic.”* We can use these tools to precisely understand logical ideas like *consistent theories* and *valid arguments*, and to show things about them.

## Key Facts

- Lots of basic logical facts (many of which will probably be familiar from your intro logic class).

## 5 The Inexpressible

### Core Skills

- We can use expressions in first-order logic to describe features of structures. This lets us understand the *expressive limits* of a theory.

## Key Facts

- *Gödel’s Fixed Point Theorem.* Any theory that can describe some basic things about strings of symbols is capable of *self-reference* (in a sense that we will make more precise).
- *Tarski’s Theorem.* No consistent theory can adequately describe itself, by expressing the property *true according to that theory*.

## 6 The Undecidable

### Core Skills

- We can use *programs* to represent general methods for answering questions.
- We can use the tools of syntax and semantics to understand programs and show things about what they can do.

### Key Facts

- *The Church-Turing Thesis*. If there is an effective method for answering a question, then we can implement this using a program.
- *Turing's Theorem*. There is no systematic method for finding out which methods will succeed—which programs will crash.
- *Church's Theorem*. There is no systematic method for finding out which sentences are consistent in first-order logic.

## 7 The Unprovable

### Core Skills

- We can use *formal proofs* to represent logically valid patterns of reasoning.

### Key Facts

- It is possible to systematically list the logical consequences of any sufficiently simple theory, one by one.
- *Gödel's First Incompleteness Theorem*. No theory is strong, simple, consistent, and complete.  
In particular, there is no complete, simple, and true theory of arithmetic—or of mathematics more generally, including logic!
- *Gödel's Second Incompleteness Theorem*. No consistent theory proves its own consistency.
- *Completeness*. Every argument in first-order logic has a proof or a counterexample, but not both.

- *Compactness*. No valid argument in first-order logic essentially relies on infinitely many premises.
- *The Löwenheim-Skolem Theorem*. Any consistent theory in first-order logic is compatible with there only being countably many things.

## 8 Second-Order Logic (if we have time)

Second-order logic adds quantifiers for “properties” or “classes.” It has a lot of important differences from first-order logic.

## Equality

This classroom is a safe space. Discrimination on the basis of race, gender, sexuality, religion, age, or other identities is unacceptable. If at any time while at USC you feel you have experienced harassment or discrimination, you can file a complaint: see <http://equity.usc.edu> for more information. You are also welcome to bring the complaint to any faculty or staff member at USC.

## Academic Conduct

Plagiarism – presenting someone else’s ideas as your own, either verbatim or recast in your own words – is a serious academic offense with serious consequences. Please familiarize yourself with the discussion of plagiarism in SCampus in Part B, Section 11, “Behavior Violating University Standards” <[policy.usc.edu/scampus-part-b](http://policy.usc.edu/scampus-part-b)>. Other forms of academic dishonesty are equally unacceptable. See additional information in SCampus and university policies on scientific misconduct, <http://policy.usc.edu/scientific-misconduct>.

## Support Systems

*Student Counseling Services (SCS) – (213) 740-7711 – 24/7 on call*

Free and confidential mental health treatment for students, including short-term psychotherapy, group counseling, stress fitness workshops, and crisis intervention. <http://engemannshc.usc.edu/counseling>

*National Suicide Prevention Lifeline – 1 (800) 273-8255*

Provides free and confidential emotional support to people in suicidal crisis or emotional distress 24 hours a day, 7 days a week. <http://www.suicidepreventionlifeline.org>

*Relationship and Sexual Violence Prevention Services (RSVP) – (213) 740-4900 – 24/7 on call*

Free and confidential therapy services, workshops, and training for situations related to gender-based harm. <http://engemannshc.usc.edu/rsvp>

*Sexual Assault Resource Center*

For more information about how to get help or help a survivor, rights, reporting options, and additional resources, visit the website: <http://sarc.usc.edu>

*Office of Equity and Diversity (OED)/Title IX Compliance – (213) 740-5086*

Works with faculty, staff, visitors, applicants, and students around issues of protected class. <http://equity.usc.edu>

*Bias Assessment Response and Support*

Incidents of bias, hate crimes and microaggressions need to be reported allowing for appropriate investigation and response. <http://studentaffairs.usc.edu/bias-assessment-response-support>

*The Office of Disability Services and Programs*

Provides certification for students with disabilities and helps arrange relevant accommodations. <http://dsp.usc.edu>

*USC Support and Advocacy (USCSA) – (213) 821-4710*

Assists students and families in resolving complex issues adversely affecting their success as a student EX: personal, financial, and academic. <http://studentaffairs.usc.edu/ssa>

*Diversity at USC*

Information on events, programs and training, the Diversity Task Force (including representatives for each school), chronology, participation, and various resources for students. <http://diversity.usc.edu>

*USC Emergency Information*

Provides safety and other updates, including ways in which instruction will be continued if an officially declared emergency makes travel to campus infeasible. <http://emergency.usc.edu>

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